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HEAT TRANSFER TO A METAL SLAB
WITH RADIATION BOUNDARY CONDITIONS

by

David C. Stickler

Contract AF 30(602)-2305
Task Number 55184

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15 October 1961

Department of ELECTRICAL ENGINEERING



THE OHIO STATE UNIVERSITY
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ABSTRACT

The plane metal slab problem has been treated in a rigorous manner for constant heat input and the 4th degree nonlinear radiation boundary condition at the front surface and an insulated rear surface. This has been possible by formulating the problem as a nonlinear Volterra integral equation. An approximate solution has been constructed with appropriate error bounds. This technique shows that the solution with the corresponding linear boundary condition yields an accurate solution for a large number of problems of practical interest. Numerical examples including rigorous estimates of their validity are given for several cases. For the case where this approach does not yield sufficiently accurate results, two alternate approaches are suggested.

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HEAT TRANSFER TO A METAL SLAB WITH RADIATION BOUNDARY CONDITIONS

PURPOSE

The problem of constant heat input to a planar metal slab with the 4th degree nonlinear radiation boundary condition at the input surface and an insulated rear wall is considered herein. The purpose of this paper is to develop an approximate solution from the rigorous formulation of the problem and to demonstrate its range of validity with appropriate error bounds. This is accomplished by formulating the problem as a nonlinear Volterra integral equation and working with the classical iterated solutions. The properties of the solution are developed from both a physical and a mathematical approach.

INTRODUCTION

In this section the problem will be stated in the customary form of a partial differential equation with boundary conditions; then physical arguments for the behavior of the solution will be presented. The arguments concerning the behavior are confirmed in Appendix II, where the same conclusions are drawn from the integral equation formulation of the problem. The integral equation formulation of the solution, which has the advantages of including the boundary conditions directly, is presented. It is based on the derivation in Appendix I. The technique for solving the integral equation is shown and the basic approximations are given.

The geometry and coordinate system of the problem are depicted in Fig. 1. The differential equation and boundary conditions are given in Eq. (1).

$$(1) \quad \frac{\partial^2 U}{\partial \xi^2} - \frac{\partial U}{\partial \tau} = 0; \quad \tau \geq 0, \quad 0 \leq \xi \leq 1$$

$$\frac{\partial U(0, \tau)}{\partial \xi} = - \frac{Q_0}{k} + \lambda U^4(0, \tau)$$

$$\frac{\partial U(l, \tau)}{\partial \xi} = 0$$

$$U(\xi, 0) = U(0, 0)$$

where $Q_0 = q_0 + \epsilon \sigma U^4(0, 0)$

$$\tau = \frac{K t}{l^2}, \quad \xi = \frac{z}{l}$$

$$\lambda = \frac{\epsilon \sigma l}{k}$$

q_0 is the heat input

ϵ is the emissivity

σ is the Stephans-Boltzman constant

t is time.

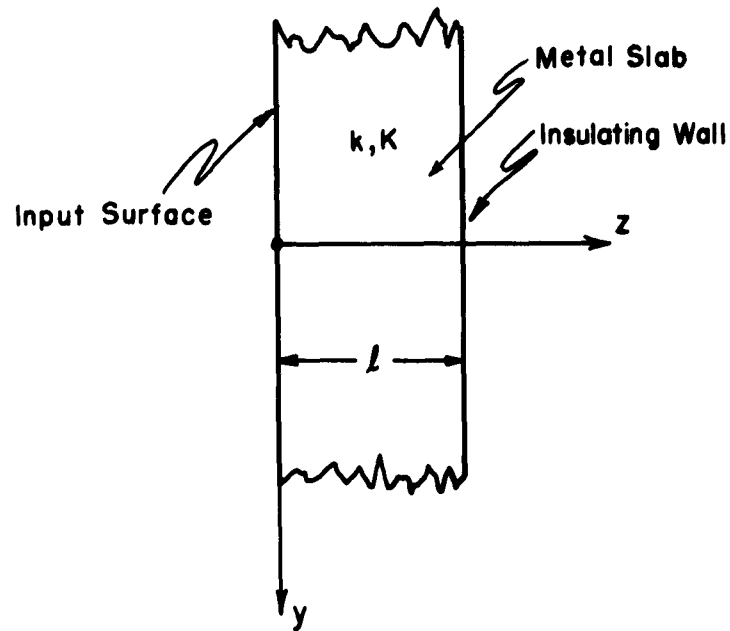


Fig. 1. Geometry of metal problem, where k is the thermal conductivity and K the diffusivity.

In order to determine the behavior of the temperature when subject to the nonlinear boundary condition, the boundary condition at the front surface is examined in detail. In Eq. (2) the boundary condition is presented in a rewritten form:

$$(2) \quad \frac{\partial U(0, \tau)}{\partial \xi} = - \frac{q_0 l}{k} + \lambda [U^4(0, \tau) - U^4(0, 0)] ,$$

where the term $\lambda [U^4(0, \tau) - U^4(0, 0)]$ represents the heat loss due to radiation. If the nonlinear term $\lambda U^4(0, \tau)$ were not present, one would expect, since the input is constant, that the temperature would increase monotonically with increasing τ and become unbounded as $\tau \rightarrow \infty$. However, with the nonlinear term present, as the temperature increases the heat input decreases due to the radiation losses, and the input term thus decreases. Hence, as before, the temperature would be expected to rise monotonically but with decreasing slope as τ is increased, and finally as $\tau \rightarrow \infty$ the input term on the right of Eq. (2) would approach zero. At this point there would be no net heat transfer; that is, the quantity of energy supplied to the slab would equal the quantity of energy lost through radiation. Thus an equilibrium temperature would be reached. This temperature will be denoted by u_e and is obtained by equating the right side of Eq. (2) to zero:

$$(3) \quad u_e = [Q_0 / \epsilon \sigma]^{1/4} .$$

Furthermore, it is expected that for $\xi \neq 0$ $U(\xi, \tau)$ will rise monotonically from $U(0, 0)$, the initial temperature (which is assumed to be less than u_e), and will approach u_e as $\tau \rightarrow \infty$. The temperature in the interior ($0 < \xi \leq 1$) will be less than the surface temperature.

Since none of the standard techniques for partial differential equations seem applicable to the solution of Eq. (1), an attempt is made to formulate Eq. (1) as an integral equation. This was begun by postulating a solution of the form

$$(4) \quad U(\xi, \tau) = U_0(\xi, \tau) - \lambda \int_{\xi'=0}^1 \int_{\tau'=0}^{\infty} G(\xi, \tau | \xi', \tau') U^4(\xi', \tau') ,$$

where $G(\xi, \tau | \xi', \tau')$ is determined by the condition that $U(\xi, \tau)$ must satisfy Eq. (1) and the boundary conditions. The term, $U_0(\xi, \tau)$, is a solution to Eq. (1) with the nonlinear boundary condition at $\xi=0$ replaced by a linear boundary condition, which can be handled by standard techniques. The boundary condition chosen was that which corresponded to the same problem but with radiation neglected, namely,

$$(5) \quad \frac{\partial U_0(0, \tau)}{\partial \xi} = - \frac{Q_0 l}{k} .$$

The solution for $U_0(\xi, \tau)$ is well known,¹ and is given by

$$(6) \quad U_0(\xi, \tau) = U(0, 0) + \frac{Q_0 l}{k} \left[\tau + \left\{ \frac{1}{3} - \xi + \frac{1}{2} \xi^2 \right\} - \frac{2}{\pi^2} \sum_{n=1}^{\infty} (n^{-2} \cos n\pi \xi) e^{-n^2 \pi^2 \tau} \right].$$

The function $G(\xi, \tau | \xi', \tau')$ was determined and details of the derivation are given in Appendix I.

$$(7) \quad U(\xi, \tau) = U_0(\xi, \tau) - \lambda \int_{\tau'=0}^{\tau} \left[1 + 2 \sum_{n=1}^{\infty} (\cos n\pi \xi) e^{-n^2 \pi^2 (\tau - \tau')} \right] U^4(0, \tau') d\tau'.$$

In the integrand of Eq. (7) only the surface temperature is unknown, thus if the surface temperature can be determined, then the temperature fields throughout the slab can be determined. An integral equation in the surface temperature is now obtained from Eq. (7) by setting $\xi = 0$. The resulting expression is given by

$$(8) \quad u(\tau) = u_0(\tau) - \lambda \int_{\tau'=0}^{\tau} g(\tau - \tau') u^4(\tau') d\tau'$$

where $u(\tau) = U(0, \tau)$

$$\begin{aligned} u_0(\tau) &= U_0(0, \tau) \\ g(\tau - \tau') &= 1 + 2 \sum_{n=1}^{\infty} e^{-n^2 \pi^2 (\tau - \tau')}, \quad \tau - \tau' > 0 \\ &= 0, \quad \tau - \tau' \leq 0. \end{aligned}$$

Before proceeding further, the relation between $u_0(\tau)$ and $g(\tau - \tau')$ will be given in order that Eq. (8) may be recast in a more illuminating form. First $u_0(\tau)$ is written in the form

$$(9) \quad u_o(\tau) = u_o(0) + \frac{Q_o l}{k} \psi_o(\tau)$$

$$\text{where} \quad \psi_o(\tau) = \tau + \frac{1}{3} - \frac{2}{\pi^2} \sum_{n=1}^{\infty} n^{-2} e^{-n^2 \pi^2 \tau},$$

and although $g(\tau - \tau')$ is not bound, it is integrable and an integration shows

$$(10) \quad \psi_o(\tau) = \int_{\tau'=0}^{\tau} g(\tau - \tau') \quad .$$

Using this relation, Eq. (8) is now recast in the form

$$(11) \quad u(\tau) = u_o(0) + \lambda \int_{\tau'=0}^{\tau} g(\tau - \tau') [u_e^4 - u^4(\tau')] \quad .$$

where u_e is given by Eq. (3). The details of this derivation are given in Appendix I. The properties of $u(\tau)$ are derived in Appendix II, and the conclusions of this work show that for $u_o(0) < u_e$,

$$(12) \quad u(\tau) \text{ is monotone increasing}$$

$$u_o(0) \leq u(\tau) < u_e$$

$$u(\tau) \xrightarrow{\tau \rightarrow \infty} u_e \quad .$$

These properties are shown pictorially in Fig. 2.

Equation (11) is solved by an iterative technique given by

$$(13a) \quad u_1(\tau) = u_o(0) + \lambda \int_{\tau'=0}^{\tau} g(\tau - \tau') [u_e^4 - u_o^4(0)]$$

$$(13b) \quad u_n(\tau) = u_o(0) + \lambda \int_{\tau'=0}^{\tau} g(\tau - \tau') [u_e^4 - u_{n-1}^4(\tau')]$$

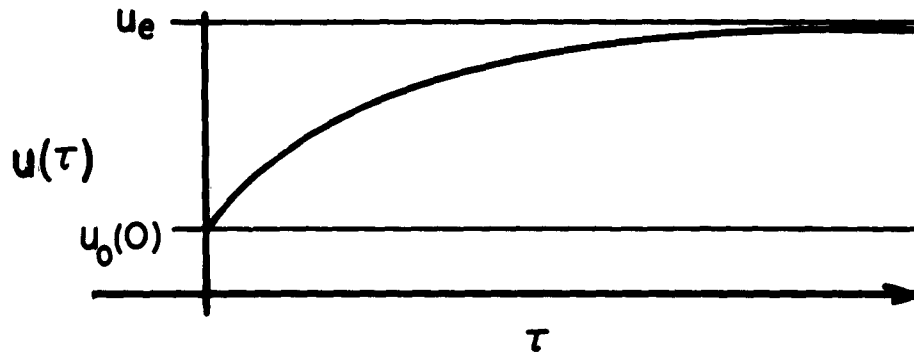


Fig. 2. Pictorial representation of $u(\tau)$, the surface temperature at $\xi = 0$, the input surface.

where it can be shown $u_n(\tau) \xrightarrow{n} u(\tau)$ under suitable restrictions. These are discussed briefly in Appendix III, but the chief concern in this report is $u_1(\tau)$ since it can be obtained explicitly and because of its relationship to $u_0(\tau)$. The computation of $u_n(\tau)$ for $n \geq 2$ cannot be carried out explicitly, and numerical methods must be employed.

However, as suggested, $u_1(\tau)$ is expressible rather simply, since $u_e^4 - u_0^4(0)$ is a constant. This can be seen from Eq. (10). Hence, it is found that $u_1(\tau)$ is given by the expression

$$(14) \quad u_1(\tau) = u_0(0) + \lambda [u_e^4 - u_0^4(0)] \psi_0(\tau) \quad .$$

The expression for $u_1(\tau)$ may also be written in the form
(since $\lambda u_e^4 = Q_0 l / k$)

$$(15) \quad u_1(\tau) = u_0(\tau) - \lambda u_0^4(0) \psi_0(\tau)$$

where $u_0(\tau)$ is given by Eq. (9) and is the solution to Eq. (1) with the nonlinear boundary condition omitted as described by Eq. (5). From Eq. (14) it is seen that $u_1(\tau)$ is monotone increasing and unbounded as $\tau \rightarrow \infty$; thus it will not be a useful approximation for all τ . However, for τ small (the precise meaning will be apparent later), $u_1(\tau)$ will be a useful approximation. In Appendix II it is shown that $u_1(\tau)$ satisfies the following inequalities:

$$(16) \quad 0 \leq u_1(\tau) - u(\tau) \leq \lambda [u_o^4(\tau) - u_o^4(0)] \psi_o(\tau).$$

In terms of $u_o(\tau)$, the solution with radiation neglected, this inequality reads, where $u_1(\tau)$ as given in Eq. (15) has been used.

$$(17) \quad \lambda u_o^4(0) \psi_o(\tau) \leq u_o(\tau) - u(\tau) \leq \lambda [u_o^4(\tau)] \psi_o(\tau).$$

The inequality in Eq. (17) indicates how close $u_o(\tau)$ is to $u(\tau)$ and will be illustrated for several examples in the next section. Thus the inequalities in Eqs. (16) and (17) define the range of τ for which the approximation is accurate, and the degree of accuracy. Numerical techniques for evaluating $\psi_o(\tau)$ and thus $u_o(\tau)$ are given in Appendix IV. Also as a result of Eq. (17) it is seen that by a slight rearrangement one obtains an upper and lower bound on $u(\tau)$, the exact solution. Explicitly this is given by the expression

$$(18) \quad u_o(\tau) - \lambda u_o^4(\tau) \psi_o(\tau) \leq u(\tau) \leq u_o(\tau) - \lambda u_o^4(0) \psi_o(\tau).$$

These upper and lower bounds on $u(\tau)$ are shown graphically for several examples in the next section. In the examples considered there the quantity $\lambda u_o^4(0) \psi_o(\tau)$ on the right of Eq. (18) was quite small and hence was neglected. Note that $u_o(\tau)$ is still an upper bound on $u(\tau)$ when $\lambda u_o^4(0) \psi_o(\tau)$ is replaced by zero.

NUMERICAL RESULTS AND EXAMPLES

In this section numerical results are given for a 1/2-inch copper slab and a 1/2-inch stainless steel slab. The results concerning the accuracy of $u_o(\tau)$ as an approximation to $u(\tau)$ are summarized in Table I when $u(\tau)$ is near the melting temperature of the slab. The errors given in the table are the largest, since the approximation improves as τ decreases. Also it is felt that there is no need to study the solution $u(\tau)$ when $u(\tau)$ exceeds the melting temperature of the slab, for surely the assumptions that the thermal parameters are independent of temperature are no longer valid in this temperature range. The thermal parameters and other data relating to the two slabs are given in Table II.

TABLE I

1/2-inch Copper Slab Melting Temperature 1325°K			1/2-inch Stainless Steel Slab Melting Temperature 1800°K		
ϵ	Max. Error	Max. Percent Error	ϵ	Max. Error	Max. Percent Error
0.03	0.02°K	< 0.001%	0.03	10°K	0.6%
1.0	175°K	< 3.8%	0.1	31°K	2%
			1.0	310°K	21%

Table I describes the accuracy of the approximate solution near the melting point of the slab.

TABLE II

$$l = 1.27 \cdot 10^{-2} \text{ meters}$$

$$u_0(0) = 300^\circ\text{K}$$

Parameter	Copper Slab		Stainless Steel Slab		
$k \frac{\text{watts}}{\text{meter}(\text{°K})}$	389		14		
$K \frac{(\text{meter})^2}{(\text{°K})(\text{sec})}$	$1.14 \cdot 10^{-4}$		$3.87 \cdot 10^{-6}$		
$Q_0 \frac{\text{watts}}{(\text{meter})^2}$	$2.8 \cdot 10^6$		$2.8 \cdot 10^6$		
ϵ	0.03	1.0	0.03	0.1	1.0
$\lambda (\text{°K})^{-3}$	$5.62 \cdot 10^{-14}$	$1.54 \cdot 10^{-12}$	$1.56 \cdot 10^{-12}$	$5.21 \cdot 10^{-12}$	$5.21 \cdot 10^{-11}$
$u_e (\text{°K})$	6500	2840	6350	4700	2640

Table II gives the parameters associated with the two slabs.

In Figs. 3 and 4 the results for the copper slab are given, but only for the case $\epsilon = 1.0$. The results for $\epsilon = 0.03$ are very good, as can be seen in Table I, and no attempt is made to graph them. The worst error is only 0.02°K and occurs when $u_0(\tau) \approx 1325^\circ\text{K}$. The results for the stainless steel slab are given in Figs. 5, 6, and 7. The worst error occurs again for $\epsilon = 1.0$.

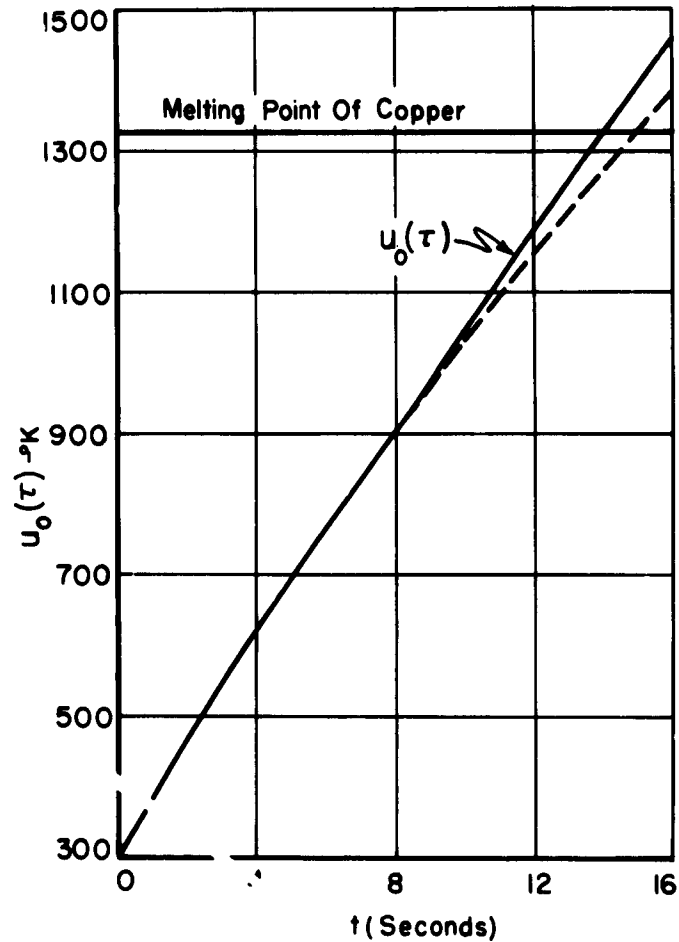


Fig. 3. Figure 3 shows $u_0(\tau)$ for the 1/2-inch copper slab. For $\epsilon = 1.0$ the exact solution, $u(\tau)$ lies above the dashed line and below $u_0(\tau)$. For $\epsilon = 0.03$ the dashed and solid curves are too close to depict separately.

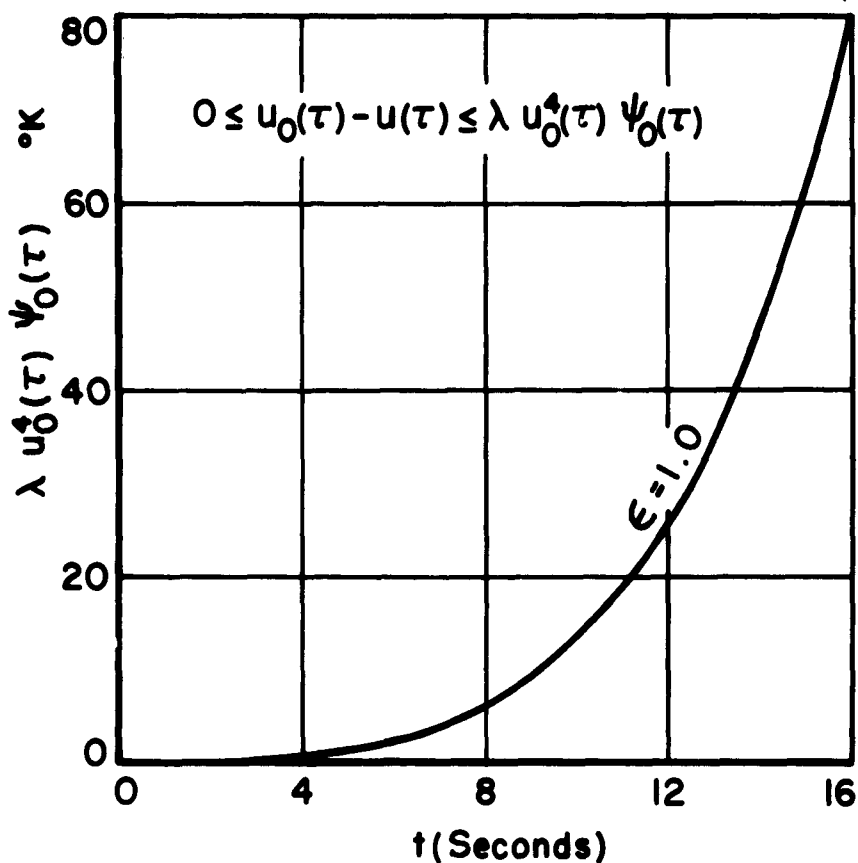


Fig. 4. Figure 4 shows the error for the 1/2-inch copper slab when $\epsilon = 1.0$. This is the worst case; that is, for all other parameters fixed, as ϵ increases the approximation becomes less accurate.

Three suggestions for improving these results will now be given. Two of the techniques will depend on a numerical integration and the third on solving a linear problem which is closer to the actual physical problem.

In deducing the upper bounds given in Eqs. (16), (17) and (18) relatively crude techniques were used, for example, since $\psi_0(\tau)$ is monotone increasing, the rather crude result follows

$$\int_{\tau=0}^{\tau} g(\tau - \tau') \psi_0(\tau') \leq \psi_0(\tau) \int_{\tau'=0}^{\tau} g(\tau - \tau') \quad .$$

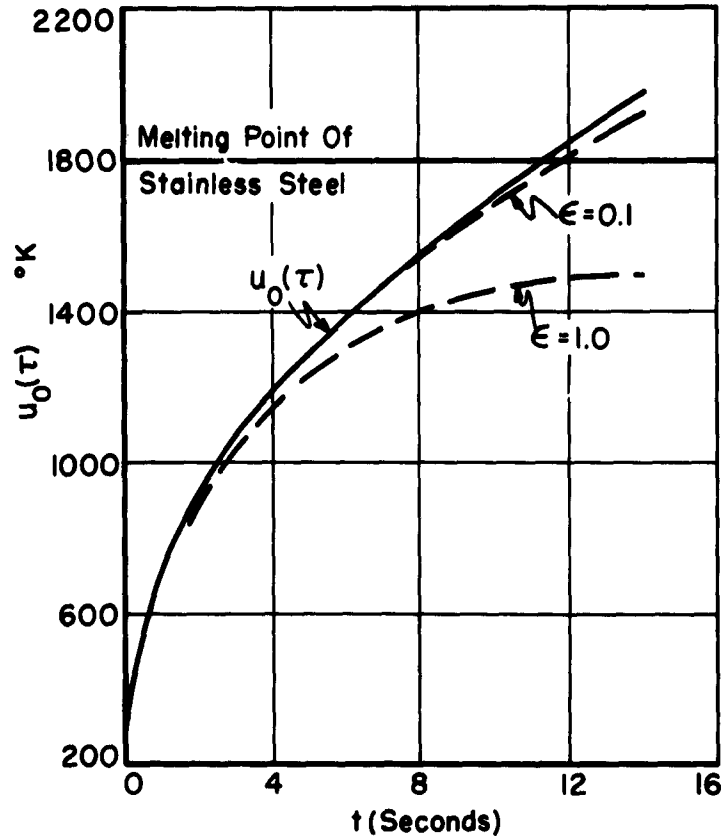


Fig. 5. Figure 5 shows $u_0(\tau)$ for the 1/2-inch stainless steel slab. For $\epsilon = 0.1$ the exact solution lies between $u_0(\tau)$ and the dashed curve labeled $\epsilon = 0.1$. For $\epsilon = 1.0$ the exact solution lies between $u_0(\tau)$ and the dashed curve labeled $\epsilon = 1.0$. For $\epsilon = 0.03$ the dashed curve is too close to $u_0(\tau)$ to depict separately on this graph.

Hence, it is quite possible that the large discrepancy indicated for the stainless steel example is an unnecessarily harsh evaluation. Therefore, the first approach would be to integrate numerically the right hand side of the inequality given in Eq. (19) below.

$$(19) \quad 0 \leq u_1(\tau) - u(\tau) \leq \lambda \int_{\tau'=0}^{\tau} g(\tau - \tau') [u_0^4(\tau') - u_0^4(0)] \quad .$$

This result is deduced from Eq. (2-6b) and the fact that $u(\tau) \leq u_0(\tau)$.

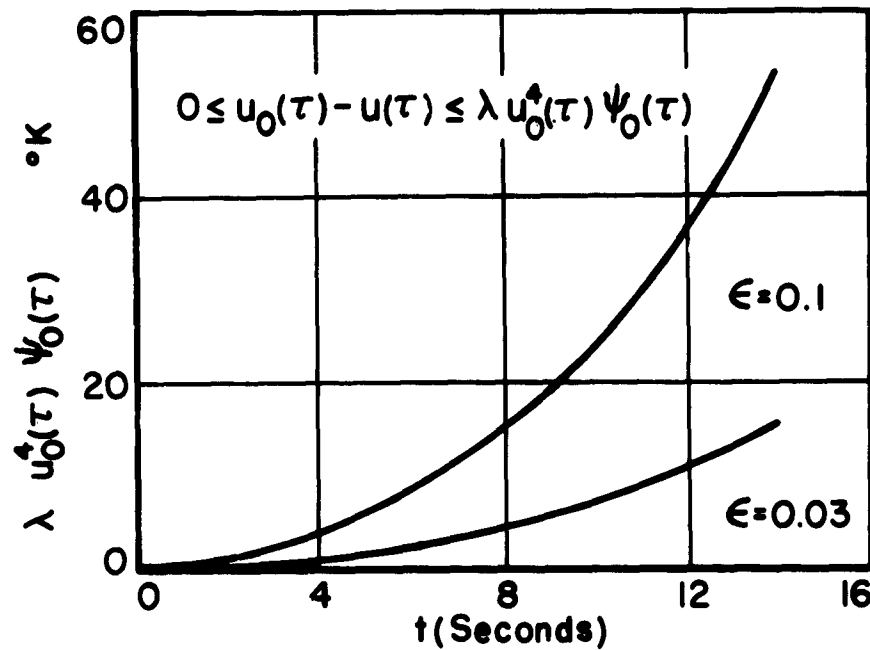


Fig. 6. Figure 6 shows the error for the 1/2-inch stainless steel slab for $\epsilon = 0.03$ and $\epsilon = 0.1$. The error for $\epsilon = 1.0$ is obtained by multiplying the ordinate by the factor 10. Thus again it is seen that as ϵ increases, so does the error.

If this approach fails to give the desired accuracy, the next step would be to compute several of the iterated solutions $u_n(\tau)$ for $n \geq 2$ and, in this example, for $0 \leq t \leq 4$ seconds. This solution for $0 \leq t \leq 4$ seconds is used to continue the solution for $t > 4$ seconds. This is accomplished by rewriting Eq. (8) as follows

$$u(\tau) = u_0(\tau) - \lambda \int_{\tau'=0}^{\tau(t=4 \text{ sec})} g(\tau - \tau') u^4(\tau') - \lambda \int_{\tau'=\tau(t=4 \text{ sec})}^{\tau} g(\tau - \tau') u^4(\tau') \quad .$$

The first two terms on the right hand side of the equality are known. Considering these two terms as the initial approximation the temperature for $t > 4$ seconds can now be computed by iteration. However, since this approach depends quite heavily on numerical integration the resultant errors from the integration may make this approach prohibitive.

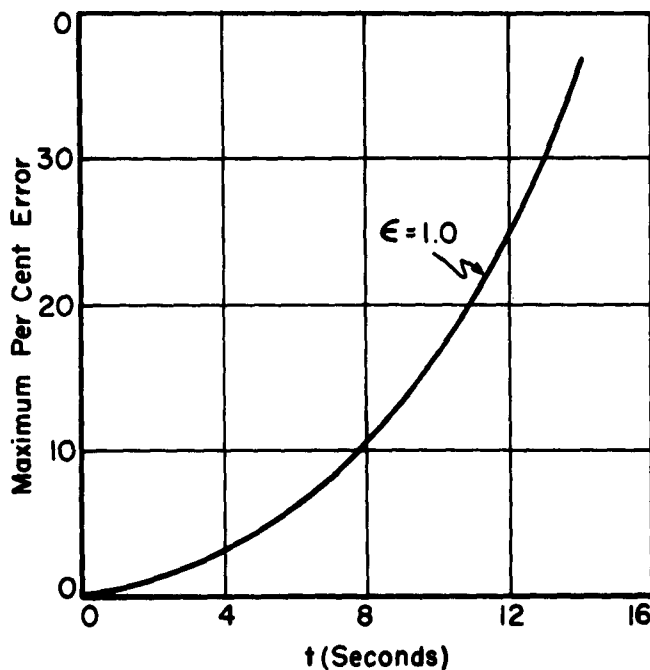


Fig. 7. Figure 7 shows the maximum percent error for the 1/2-inch stainless steel slab when $\epsilon = 1.0$, the worst case.

The third approach, as mentioned earlier, consists of solving a linear problem which is closer to the physical problem and evaluating its accuracy by the techniques presented here. The first step is to expand the right hand side of Eq. (2) in a power series about $U(0,0)$. This is given by

$$(20) \quad \frac{\partial U}{\partial \xi}(0, \tau) = -\frac{q_0 l}{k} + 4\lambda U^3(0,0)[U-U(0,0)] + \lambda H(U)$$

where
$$H(U) = 6 U^2(0,0)[U-U(0,0)]^2 + 4U(0,0)[U-U(0,0)]^3 + [U-U(0,0)]^4.$$

It is seen that the right hand side of Eq. (20) involves terms which are linear in U and terms which are nonlinear in U . The nonlinear terms are grouped in the factor $H(U)$. It is possible to obtain a useful solution to Eq. (1) when the nonlinear terms are neglected. This solution has two advantages over linear solution which has been used in the examples considered here.

- (1) it takes into account some radiation loss for τ small while the previous did not.
- (2) this solution will be bounded as $\tau \rightarrow \infty$, while the previous was not.

Also, the nonlinear term may be expanded in a power series about the equilibrium temperature. This will yield an approximate solution which will be useful for τ large.

No numerical results based on this approach are available but of the three methods suggested here this appears to be the most promising.

CONCLUSIONS

The metal slab problem with the nonlinear boundary conditions has been formulated as a nonlinear Volterra integral equation. A simple approximate solution has been found which is shown to be quite adequate for several cases of practical interest. Error bounds indicating the validity of the approximation have been given. Techniques have been suggested for handling the case when the error becomes too large.

Since it has been possible to determine the equilibrium temperature by examining the integral equation satisfied by the surface temperature, the possibility exists that the equilibrium temperature for more complex shapes can be determined by a similar procedure.

APPENDIX I

In this section an argument is given for reformulation Eq. (1) as an integral equation. A similar argument based on Green's second theorem can be given. The equations to which a solution is sought are restated in Eq. (AI-1a) to Eq. (AI-1d).

$$(AI-1a) \quad \frac{\partial^2 U}{\partial \xi^2} - \frac{\partial U}{\partial \tau} = 0, \quad \tau \geq 0, \quad 0 \leq \xi \leq 1,$$

$$(AI-1b) \quad \frac{\partial U}{\partial \xi}(0, \tau) = -\frac{Q_0 l}{k} + \lambda U^4(0, \tau),$$

$$(AI-1c) \quad \frac{\partial U}{\partial \xi}(1, \tau) = 0,$$

$$(AI-1d) \quad U(\xi, 0) = U(0, 0),$$

where the symbols are defined in Eq. (1). An attempt is made to seek an integral equation formulation of the problem by asking for a representation of $U(\xi, \tau)$ in the form

$$(AI-2) \quad U(\xi, \tau) = U_0(\xi, \tau) - \lambda \int_{\tau=0}^{\infty} \int_{\xi'=0}^1 G(\xi, \tau | \xi', \tau') U^4(\xi', \tau') d\xi' d\tau',$$

where $U_0(\xi, \tau)$ satisfies Eq. (AI-1) with $\lambda = 0$, and $G(\xi, \tau | \xi', \tau')$ is determined by the requirement that $U(\xi, \tau)$ satisfy Eqs. (AI-1). The solution for $U_0(\xi, \tau)$ is well known¹ and is given in Eq. (6).

The requirement that $U(\xi, \tau)$ satisfy Eq. (AI-1a) requires that $G(\xi, \tau | \xi', \tau')$ satisfy

$$\frac{\partial^2 G}{\partial \xi^2}(\xi, \tau | \xi', \tau') - \frac{\partial G}{\partial \tau}(\xi, \tau | \xi', \tau') = 0$$

$$\begin{array}{ll} \tau \geq 0 & 0 \leq \xi \leq 1 \\ \tau' \geq 0 & 0 \leq \xi' \leq 1 \end{array}.$$

Notice that $U_0(\xi, \tau)$ satisfies Eq. (AI-1a). Saving the boundary condition in Eq. (AI-1b) until last, the boundary condition in Eq. (AI-1c) is considered next. First note that

$$\frac{\partial U_0}{\partial \xi} (1, \tau) = 0 \quad ,$$

and hence, in order that $U(\xi, \tau)$ satisfy Eq. (AI-1c), it is required that

$$\frac{\partial G}{\partial \xi} (1, \tau | \xi', \tau') = 0 \quad .$$

Turning next to the initial condition in Eq. (AI-1d), the following conclusions are drawn. Since

$$U_0(\xi, 0) = U(0, 0) \quad ,$$

it is required that

$$G(\xi, 0 | \xi', \tau') = 0 \quad .$$

Now attention is turned to the nonlinear boundary condition expressed in Eq. (AI-1b). Next, it is noted that

$$\frac{\partial U_0}{\partial \xi} (0, \tau) = - \frac{Q_0 l}{k} \quad .$$

Hence, it is seen that if $G(\xi, \tau | \xi', \tau')$ satisfies

$$\frac{\partial G}{\partial \xi} (0, \tau | \xi', \tau') = -\delta(\xi') \delta(\tau - \tau') \quad ,$$

where $\delta(x)$ is the Dirac delta function, then $U(\xi, \tau)$ satisfies the boundary condition at the input surface. Here use has been made of the fact that

$$U^4(0, \tau) = \int_{\tau'=0}^{\infty} \int_{\xi'=0}^1 \delta(\xi') \delta(\tau - \tau') U^4(\xi', \tau') \quad .$$

In summary, then, it is seen that $G(\xi, \tau | \xi', \tau')$ satisfies the following system of equations:

$$(AI-3) \quad \frac{\partial^2 G}{\partial \xi^2} - \frac{\partial G}{\partial \tau} = 0 \quad 0 \leq \xi \leq 1, 0 \leq \tau$$

$$\frac{\partial G}{\partial \xi} (0, \tau | \xi', \tau') = -\delta(\xi') \delta(\tau - \tau')$$

$$\frac{\partial G}{\partial \xi} (1, \tau | \xi', \tau') = 0, \quad \tau \geq 0$$

$$G(\xi, 0 | \xi', \tau') = 0, \quad 0 \leq \xi \leq 1,$$

$$\text{for} \quad 0 \leq \xi' \leq 1 \quad 0 \leq \tau'.$$

The solution to this system (Eq. (AI-3)) is quite similar to that for $U_0(\xi, \tau)$, as can be seen by comparison. The solution for $G(\xi, \tau | \xi', \tau')$ is given by

$$(AI-4) \quad G(\xi, \tau | \xi', \tau') = \delta(\xi') \left[1 + 2 \sum_{n=1}^{\infty} [\cos n\pi \xi] e^{-n^2 \pi^2 (\tau - \tau')} \right] s(\tau - \tau')$$

$$\text{where} \quad s(\tau - \tau') = 0, \quad \tau - \tau' \leq 0 \\ = 1, \quad \tau - \tau' > 0.$$

The formal solution for $G(\xi, \tau | \xi', \tau')$ was obtained by the Laplace Transform techniques. Substituting this expression (Eq. (AI-4)) into Eq. (AI-2) and carrying out the integration with respect to ξ' , one obtains the result in Eq. (7) of Section B, namely,

$$(AI-5) \quad U(\xi, \tau) = U_0(\xi, \tau) - \lambda \int_{\tau'=0}^{\tau} \left[1 + 2 \sum_{n=1}^{\infty} (\cos n\pi \xi) e^{-n^2 \pi^2 (\tau - \tau')} \right] U^4(0, \tau')$$

$$g(\xi, \tau - \tau') = 1 + 2 \sum_{n=1}^{\infty} \cos n\pi \xi e^{-n^2 \pi^2 (\tau - \tau')} \quad \tau - \tau' > 0$$

$$= 0 \quad \tau - \tau' \leq 0.$$

It has not been possible to show in a rigorous fashion that $U(\xi, \tau)$ is determined by Eq. (AI-5); however, the argument appears reasonable, and as will be seen in the next section, the temperature, $U(\xi, \tau)$, has all of the expected physical properties.

APPENDIX II

In this section the properties of $u(\tau)$, the surface temperature, are established. These properties are deduced by assuming that $u(\tau)$ is bounded and integrable and $u_0(0) < u_e$. The case $u_0(0) = u_e$ has the trivial solution $u(\tau) = u_e$ and the case $u_0(0) > u_e$ is of no interest physically, although it can be handled in the same manner as the present case.

From Eq. (AI-5) in Appendix I, by setting $\xi = 0$ one obtains the following expression:

$$(AII-1) \quad u(\tau) = u_0(\tau) - \lambda \int_{\tau'=0}^{\tau} g(\tau - \tau') u^4(\tau') d\tau'$$

where $u(\tau) = U(0, \tau)$

$$u_0(\tau) = U_0(0, \tau)$$

$$g(\tau - \tau') = 1 + 2 \sum_{n=1}^{\infty} e^{-n^2 \pi^2 (\tau - \tau')} , \quad \tau - \tau' > 0$$

$$= 0 \quad \tau - \tau' \leq 0 .$$

Note that Eq. (AII-1) implies that $u_0(\tau) \geq u(\tau)$, $\tau \geq 0$. Elementary theorems² show that $g(\tau - \tau')$ is summable and that

$$(AII-2) \quad \psi_0(\tau) = \int_{\tau'=0}^{\tau} g(\tau - \tau') d\tau' .$$

With this information and the relation between $\psi_0(\tau)$ and $u_0(\tau)$, one deduces the integral equation given in Eq. (11),

$$(AII-3) \quad u(\tau) = u_0(0) + \lambda \int_{\tau'=0}^{\tau} g(\tau - \tau') [u_e^4 - u^4(\tau')] d\tau'$$

where $u_e^4 = \frac{Q_0}{\epsilon \sigma} = \frac{Q_0 l}{\lambda k} .$

Several elementary observations are now made.

$$u(0) = u_0(0),$$

and there exists a τ^* such that $u(\tau)$ is continuous for $0 \leq \tau < \tau^*$. Since $u(\tau)$ is continuous for $\tau < \tau^*$ and $u(0) = u_0(0) < u_e$, it follows that $u(\tau) < u_e$ for some interval to the right of $\tau = 0$. Thus, it also follows, since the integrand in Eq. (AII-3) is positive, that $u(\tau)$ is monotone increasing in the same interval that for which $u(\tau) < u_e$. Thus it has been established that

$$u(\tau) \text{ is monotone increasing } \tau < \tau_0$$

$$\text{where } u(\tau_0) = u_e.$$

It will now be established that τ_0 is not finite. For $\tau > \tau_0$, examination of Eq. (AII-3) shows that

$$(AII-4) \quad u(\tau) = u_0(0) + \lambda \int_{\tau'=0}^{\tau_0} g(\tau-\tau') [u_e^4 - u^4(\tau')] \\ + \lambda \int_{\tau'=\tau_0}^{\tau} g(\tau-\tau') [u_e^4 - u^4(\tau')] \quad .$$

Since $\tau > \tau_0$, it follows that $g(\tau_0 - \tau') > g(\tau - \tau')$. Hence, since $u(\tau') < u_e$ for $\tau' < \tau_0$,

$$(AII-5) \quad u(\tau) \leq u_e + \lambda \int_{\tau'=\tau_0}^{\tau} g(\tau-\tau') [u_e^4 - u^4(\tau')], \quad \tau > \tau_0 \quad .$$

But it is assumed that $u^4(\tau') > u_e^4$ for $\tau' > \tau_0$, which implies $u(\tau) < u_e$; this contradicts the assumption that $u(\tau)$ equals u_e for finite τ . Thus $u(\tau)$ is a monotone increasing function for $\tau \geq 0$, bounded below by $u_0(0)$ and above by u_e , and further

$$u(\tau) \xrightarrow{\tau \rightarrow \infty} u_e \quad .$$

These are precisely the results based on physical reasoning.

Next the bounds on the difference, $u_1(\tau) - u(\tau)$, will be derived. From Eq. (AII-3) and the expression for $u_1(\tau)$ given in Eq. (AII-6a), the following results are deduced:

$$(AII-6a) \quad u_1(\tau) = u_0(0) + \lambda \int_{\tau'=0}^{\tau} g(\tau - \tau') [u_e^4 - u_0^4(0)] \, d\tau'.$$

Hence, it is seen that (since $u(\tau) > u_0(0)$)

$$(AII-6b) \quad 0 \leq u_1(\tau) - u(\tau) = \lambda \int_{\tau'=0}^{\tau} g(\tau - \tau') [u^4(\tau') - u_0^4(0)] \, d\tau'.$$

From the fact that $u^4(\tau)$ is monotone increasing, and bounded by the monotone increasing function $u_0^4(\tau)$, it follows that

$$(AII-7) \quad 0 \leq u_1(\tau) - u(\tau) \leq \lambda [u_0^4(\tau) - u_0^4(0)] \psi_0(\tau).$$

This is the result stated in Eq. (16). Thus the properties of $u(\tau)$ as well as the relationship between $u(\tau)$, $u_1(\tau)$ and $u_0(\tau)$ have been deduced. Some of the properties of $u_n(\tau)$ for $n \geq 2$ are deduced in Appendix III, although they are not of particular interest as far as this report is concerned.

APPENDIX III

As pointed out previously, since $u_1(\tau)$ is unbounded as $\tau \rightarrow \infty$, it surely will not offer a good approximation to $u(\tau)$ for all τ . This is also illustrated in an example in Section 2. It is therefore the purpose of this section to show how the iterated solutions, $u_n(\tau)$, $n \geq 2$, can be used to improve the approximation. In this section it will be shown that for τ less than a certain τ_0 , the convergence of $u_n(\tau)$ to $u(\tau)$ is guaranteed. For τ larger than this value another method is suggested. Let τ_1 be defined as follows:

$$(AIII-1) \quad u_1(\tau_1) = u_e .$$

The next iterated solution $u_2(\tau)$ is given by (as defined by Eq. (13b))

$$u_2(\tau) = u_0(0) + \lambda \int_{\tau'=0}^{\tau} g(\tau-\tau') [u_0^4 - u_1^4(\tau')] .$$

Then for $\tau < \tau_1$,

$$u_0(0) \leq u_2(\tau) \text{ and is monotone increasing.}$$

Further, for $\tau < \tau_1$

$$u_2(\tau) \leq u_1(\tau) \text{ and}$$

$$u_0(0) \leq u_2(\tau) \leq u(\tau) \leq u_1(\tau)$$

with the equality signs holding for $\tau = 0$ only. To show that $u(\tau) \geq u_2(\tau)$, consider the difference

$$u(\tau) - u_2(\tau) = \lambda \int_{\tau'=0}^{\tau} g(\tau-\tau') [u_1^4(\tau') - u^4(\tau')] .$$

Since $u_1(\tau) \geq u(\tau)$, $\tau \leq \tau_1$ the result follows. In a similar manner one can show

$$u(\tau) \leq u_3(\tau) \leq u_1(\tau), \tau < \tau_1.$$

In general one has the following scheme with each $u_n(\tau)$ a monotone increasing function of τ for $\tau < \tau_1$.

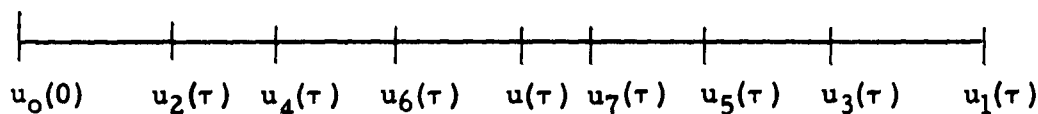


Fig. III-1.

In this figure a pictorial representation of the type of convergence is shown. The question to ask is, for what range of τ does the sequence $|u(\tau) - u_n(\tau)|$ converge? This sequence can be shown to converge for all τ which satisfy

$$(AIII-2) \quad 4\lambda \psi_0(\tau) u_1^3(\tau) < 1.$$

Denote the largest least upper bound of the τ 's which satisfy Eq. (AIII-2) by τ_2 ; then for

$$(AIII-3) \quad \tau < \min [\tau_1, \tau_2] \quad \text{the sequence } u_n(\tau) \xrightarrow{n \rightarrow \infty} u(\tau).$$

For $\tau > \tau_0$ where $\tau_0 < \min [\tau_1, \tau_2]$, another technique must be sought. Let τ_0 be a value for which an accurate iterated solution has been found. Rewrite the integral equation for $u(\tau)$ as follows:

$$u(\tau) = u_0(0) + \lambda \int_{\tau'=0}^{\tau_0} g(\tau - \tau') [u_e^4 - u^4(\tau')] + \lambda \int_{\tau'=\tau_0}^{\tau} g(\tau - \tau') [u_e^4 - u^4(\tau')]$$

for $\tau > \tau_0$.

Note that the term

$$u_0(0) + \lambda \int_{\tau'=0}^{\tau_0} g(\tau - \tau') [u_e^4 - u^4(\tau')] d\tau'$$

is known, and hence for $\tau > \tau_0$ one is faced with solving the integral equation

$$u(\tau) = f(\tau) + \lambda \int_{\tau'=\tau_0}^{\tau} g(\tau - \tau') [u_e^4 - u^4(\tau')] d\tau' .$$

It is now hoped that this equation can be iterated to obtain approximate solutions for $\tau > \tau_0$. However, this matter will not be pursued further here.

APPENDIX IV

In this section some information concerning the calculation of $\psi_0(\tau)$ is given. This function is given by

$$(AIV-1) \quad \psi_0(\tau) = \tau + \frac{1}{3} - \frac{2}{\pi^2} \sum_{n=1}^{\infty} n^{-2} e^{-n^2 \pi^2 \tau},$$

and due to the exponential terms in the series one expects good convergence for τ large. This is indeed the case as the following tables indicate. However, for τ small the convergence is slow and another form for the expression of $\psi_0(\tau)$ is sought. This expression and a resulting simple approximating function is found by noting that $g(\tau - \tau')$ is a Theta function. Relationships between the Theta functions yield the desired relationship.³

$$(AIV-2) \quad g(\tau - \tau') = \theta_3(0, |i\pi(\tau - \tau')|)$$

$$= \frac{1}{\sqrt{\pi(\tau - \tau')}} \theta_3\left(0, \left| \frac{i}{\pi(\tau - \tau')} \right| \right)$$

where $\theta_3(0, |ix|) =$

$$1 + 2 \sum_{n=1}^{\infty} e^{-n^2 \pi x}.$$

The following results were then deduced, and are given in Tables III, IV, and V. Table III gives the general results, Table IV gives results when accuracy to five places is needed, and Table V gives results only when three-place accuracy is needed.

TABLE III

Various approximate expressions for $\psi_0(\tau)$ and bounds denoting how close the bounds are.

$0 \leq \psi_0(\tau) - 2\sqrt{\frac{\tau}{\pi}} \leq R$ $R = \frac{1}{3} (\pi \tau)^{3/2} e^{-1/\tau}$
$0 \leq \tau + \frac{1}{3} - \frac{2}{\pi^2} (e^{-\pi^2 \tau} + \frac{1}{4} e^{-4\pi^2 \tau} + \frac{1}{9} e^{-9\pi^2 \tau}) - \psi_0(\tau) \leq R$ $R = 5.75 \cdot 10^{-2} e^{-16\pi^2 \tau}$
$0 \leq \tau + \frac{1}{3} - \frac{2}{\pi^2} (e^{-\pi^2 \tau} + \frac{1}{4} e^{-4\pi^2 \tau}) - \psi_0(\tau) \leq R$ $R = 8 \cdot 10^{-2} e^{-9\pi^2 \tau}$
$0 \leq \tau + \frac{1}{3} - \frac{2}{\pi^2} (e^{-\pi^2 \tau}) - \psi_0(\tau) \leq R$ $R = .131 e^{-4\pi^2 \tau}$
$0 \leq \tau + \frac{1}{3} - \psi_0(\tau) \leq R$ $R = \frac{1}{3} e^{-\pi^2 \tau}$

TABLE IV

Gives approximate expressions for $\psi_0(\tau)$
accurate to five significant figures.

Range of τ	Approximate Expression $\psi_0(\tau)$ Five significant figures
$0 \leq \tau \leq .1$	$\psi_0(\tau) = 2 \sqrt{\frac{\tau}{\pi}}$
$.1 \leq \tau \leq .15$	$\psi_0(\tau) = \tau + \frac{1}{3} - \frac{2}{\pi^2} (e^{-\pi^2 \tau} + \frac{1}{4} e^{-4\pi^2 \tau} + \frac{1}{9} e^{-9\pi^2 \tau})$
$.15 \leq \tau \leq .30$	$\psi_0(\tau) = \tau + \frac{1}{3} - \frac{2}{\pi^2} (e^{-\pi^2 \tau} + \frac{1}{4} e^{-4\pi^2 \tau})$
$.30 \leq \tau \leq 1.25$	$\psi_0(\tau) = \tau + \frac{1}{3} - \frac{2}{\pi^2} (e^{-\pi^2 \tau})$
$1.25 \leq \tau$	$\psi_0(\tau) = \tau + \frac{1}{3}$

TABLE V

Gives approximate expressions for $\psi_0(\tau)$
accurate to three significant figures.

Range of τ	Approximate Expression $\psi_0(\tau)$ Three significant figures
$0 \leq \tau \leq .1$	$\psi_0(\tau) = 2 \sqrt{\frac{\tau}{\pi}}$
$.1 \leq \tau \leq .2$	$\psi_0(\tau) = (\tau + \frac{1}{3}) - \frac{2}{\pi^2} (e^{-\pi^2 \tau} + \frac{1}{4} e^{-4\pi^2 \tau})$
$.2 \leq \tau \leq .7$	$\psi_0(\tau) = (\tau + \frac{1}{3}) - \frac{2}{\pi^2} (e^{-\pi^2 \tau})$
$.7 \leq \tau$	$\psi_0(\tau) = \tau + \frac{1}{3}$

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